Solving Differential Equations with Spreadsheets

I. Objective

To use spreadsheets to solve differential equations of the form $\frac{dx}{dt} = f(t,x)$.

II. Introduction

Spreadsheets are very useful for implementing **recursive formulas**, i.e. formulas that determine the value of a function in terms of its value at a previous point. Today, we will see how algorithms for solving differential equations may be expressed in terms of recursion formulas and therefore easily solved in a spreadsheet.

A very important class of differential equations, called **first order ordinary differential equations**, is of the form $\frac{dx}{dt} = f(t,x)$, where t is the independent variable, x is the dependent variable, and f is a function of t and x. Recall from your study of first order differential equations that one must also know the value x(t) at some time (usually taken to be t=0) in order to obtain a unique solution. Therefore, in our work today, we must be given x(0) in order to numerically solve the differential equation.

After determining the numerical solution of a differential equation, it is very important to critically study the result. *Assume your result is wrong until proven correct.* There are several ways that the answer may be checked:

- **Compare your result to what you expect based on your physical intuition**: Based on the physics of the problem, what do you expect to happen at zero? at infinity? Do you expect the function to be positive or negative? Should it go to zero? Try to understand the physics of the problem and beware of results that are inconsistent with your understanding.
- **Compare your result to a simple special case:** If you are trying to solve a complicated problem such as
$$\frac{dx}{dt} = Ax^2 + Bx + C$$
you might check your solution for the special case *A=0* , *B=0.* The resulting solution is *x(t)=Ct+D,* which is easy to check because the graph is a straight line.
- **Compare your result to results obtained by another algorithm:** This is self explanatory. However, in cases where you do not have another algorithm, an alternative is to observe the effect of your algorithm on changing the step size. If the solution does not change, you may gain confidence in your result.

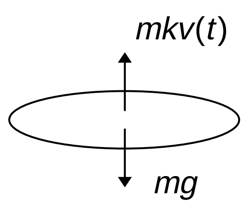
We will apply each of the above methods in the exercises.

III. Exercises

A. Solving a Differential Equation and Checking the Results

Objective:	to numerically solve the equation of motion for a falling object with air resistance
Where to begin:	start in MS Excel
What to do:	create a spreadsheet that solves the differential equation $\frac{dv(t)}{dt} = g - kv(t)$ with boundary condition v(0) = 0 over the range t=[0,12 s]
What to turn in to your instructor:	(1) graphs described below; (2) your comments on the methods you used to check your results
What to put in log book:	the time you begin your work, problems, solutions, new commands, etc.

(1) **Problem:** Suppose that a sky diver falls vertically from some initial position with only the forces of gravity and air resistance acting on him. Suppose further that air resistance produces a drag force proportional to the sky diver's speed. Newton's second law ($ma_y = \sum F_y$) leads to the equation of motion



$$\frac{dv(t)}{dt} = g - kv(t)$$

where $v(t)$ is the speed at time t, g is the acceleration of gravity, m is the mass of the skydiver (it occurs in the force diagram above but cancels out of the equation of motion), and k is the drag constant.

(2) **Spreadsheet:** Create a spreadsheet that solves the above equation of motion numerically for $v(t)$ using the Euler Method (see the Appendix) and graphs $v(t)$ versus t. You should give the spread sheet the following parameters: ***g*=9.80 m/s^2, *v(0)*=0 m/s, *k*=0.5 s^{-1}, and step size *h*=0.1 s**. You should set up your **time axis to run from t=0 to t=12 s** in steps of h=0.1 s. We suggest the layout indicated below with the graph positioned so that you can change the parameters $v(0)$, k, and h and see the graph of your results.

(3) **Compare your results with physical intuition:** Now you must critically examine the behavior of the numerical solution in order to gain confidence in the result. Does your graph of $v(t)$ make physical sense? Why? Give a *quantitative* explanation of why you think your numerical solution is correct based on the value of g and k you gave the spreadsheet (hint: compare the value for the terminal velocity you expect from the physics with the value given by the graph of $v(t)$).

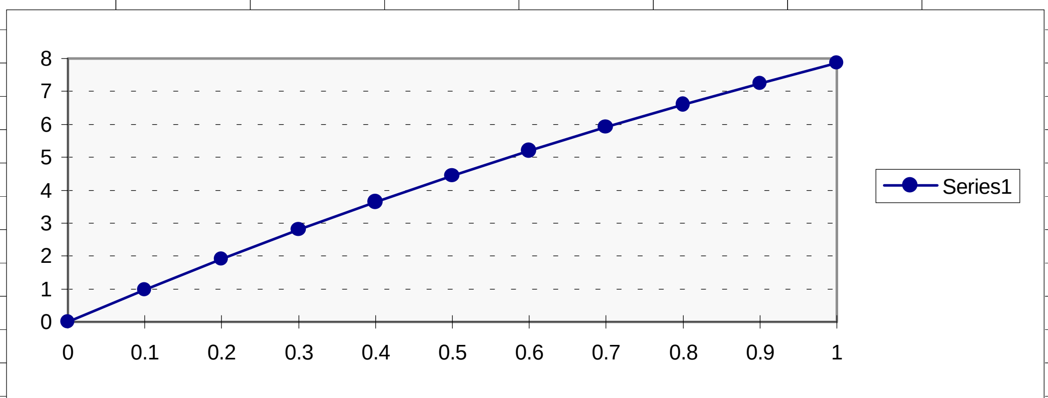
(4) **Compare your results with a special case of the solution:** Another way to check the result is to solve a simple special case. What value of the parameter k would you choose to simplify the equation of motion? Does the graph of the numerical solution for this case agree with what you expect based on the physics you learned in Physics 221? Explain your method and comment on your results.

(5) **Compare results with those of another algorithm:** Still another check on your solution is to compare it with the solution given by another algorithm. Solve the equation of motion of the skydiver using the Runge-Kutta Method (see Appendix). Does this solution agree with your previous one? Print out the graph of both the Euler and Runge-Kutta solutions (no data please) and comment on their agreement.

(6) **Save your work:** Save your spreadsheet on your diskette and call it **deq.xls**.

(7) **Optional:** If you have some time left at the end, solve this differential equation also with the *Runge-Kutta* method described in the appendix and compare with your results using the Euler method.

Skydiver			
v(0)	**g**	**k**	**step h**
0	9.8	0.5	0.1
t	**v**		
0	0		
0.1	0.98		
0.2	1.911		
0.3	2.79545		
0.4	3.635678		
0.5	4.433894		
0.6	5.192199		
0.7	5.912589		
0.8	6.59696		
0.9	7.247112		
1	7.864756		



B. Solving a Set of Coupled Differential Equations

Objective:	to develop the recursion formula for solving two coupled differential equations using Euler's Method
Where to begin:	in your log book
What to do:	follow the discussion below to develop a formula for numerically solving the set of coupled equations $$\frac{dx}{dt} = f(t,x,y)$$ $$\frac{dy}{dt} = g(t,x,y)$$ using the Euler Method.
What to turn in to your instructor:	your derivation
What to put in log book:	your derivation

(1) **The Euler Method for a Single Differential Equation:** Euler's Method is the simplest and most intuitive formula for numerically solving differential equations. In the simplest case one has only one differential equation given by

$$\frac{dx}{dt} = f(t, x)$$

where t is the independent variable and x is the dependent variable. The Euler Method for solving the above differential equation is obtained by approximating the derivative at time $t=t_n$ by the numerical derivative:

$$\frac{dx}{dt} \to \frac{x_{n+1} - x_n}{t_{n+1} - t_n}$$

Therefore

$$\frac{x_{n+1} - x_n}{h} = f(t_n, x_n)$$

where we have defined the step size $h \equiv t_{n+1} - t_n$. Solving for x_{n+1} therefore gives

$$x_{n+1} = x_n + hf(t_n, x_n)$$

which is the recursion relation given in the Appendix for Euler's Method.

(2) **Deriving Euler's Method for Two Coupled Differential Equations:** Many very interesting problems in physics result in coupled differential equations of the form

$$\frac{dx}{dt} = f(t, x, y)$$

$$\frac{dy}{dt} = g(t, x, y)$$

(the term **coupled** means "connected" and refers to the fact that the x value helps determine the y value, and vice versa).

Follow the procedure in part **(1)** and derive a set of two coupled recursive formulas that would allow one to numerically solve the coupled differential equations above. Write your derivation in your log book.

(3) **Solving a Coupled System (Optional):** For fun you may wish to try out your new formula. Set up a spreadsheet to numerically solve the equations

$$\frac{dx}{dt} = -y$$

$$\frac{dy}{dt} = x$$

given the initial conditions $x(0)=1$, $y(0)=0$ from t=[0,12 s]. Note that the exact solution to the above equations is $x=\cos(t)$ and $y=\sin(t)$. Graph both x(t) and y(t) on the same graph. Play around with different step sizes and watch what happens. Observe and record how your numerical result deviates from the exact result as you try to extend the solution to times longer than t=12 s.

IV. Appendix: Solving Differential Equations[1]

The table below gives recursive formulas for solving differential equations of the form

$$\frac{dx}{dt} = f(t,x)$$

given the value of x at t =0. We are looking for a function $x(t)$, which satisfies this equation. The symbol $f(t,x)$ denotes a function of t (the independent variable) and x (the dependent variable). We define the step size $h \equiv t_{n+1} - t_n$ and the symbol $x_n \equiv x(t_n)$. The term **global error** in the table refers to the error in x_n at the final t value. The notation O(h) means that the error is on the order of the value h.

Method Name	Formula	Global Error	Reference
Euler's Method	$x_{n+1} = x_n + hf(t_n, x_n)$	O(h)	Koonin p.6
Runge-Kutta Method[2]	$x_{n+1} = x_n + hf(t_n + \frac{h}{2}, x_n + \frac{r}{2})$ $r = hf(t_n, x_n)$	$O(h^2)$	Koonin p.6

[1] These formulas are discussed in the book *Computational Physics* by Steven Koonin which is on reserve in Parks Library.

[2] There are many different Runge-Kutta algorithms. See Koonin for methods of order 3 and higher.